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On function spaces topologies in the setting of Čech closure spaces

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ABSTRACT

We consider different types of topologies on the set of functions between two Čech closure spaces and investigate some of their properties.

1. Introduction

In the classic paper of Arens and Dugundji [2] an exhaustive study of proper and admissible topologies on the set of continuous functions was undertaken. Naimpally in [16] introduced the notion of graph topology and investigated some of its properties. This topology was further studied in [17,4,12] and some other papers.

In [5], besides other topics, θ-continuous functions from a space X to a space Y were discussed and a topology on the set of these functions analogous to the compact-open topology. A special Čech closure operator, θ-closure, was considered in [14].

In [8–10] different types of continuous-like functions (defined by means of θ-closure) between topological spaces have been considered and topologies on sets of these functions investigated. It was shown in [13] that some of the obtained results for function spaces hold in the setting of closure spaces as well, in particular those concerning proper and admissible topologies.

Kočinac in [11] studied some closure properties of the space $C_k(X)$ of continuous real-valued functions on a space X equipped with the compact-open topology.

The aim of this paper is to generalize the notions of the compact-open and graph topology to the set of functions between two Čech closure spaces and to investigate some properties of these spaces. In particular, the separation properties of the initial spaces are related to those of function spaces.

2. Preliminaries

First we recall several definitions.

An operator $u : \mathcal{P}(X) \to \mathcal{P}(X)$ defined on the power set $\mathcal{P}(X)$ of a set X satisfying the axioms:

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(C1) $u(\emptyset) = \emptyset$,
(C2) $A \subset u(A)$ for every $A \subset X$,
(C3) $u(A \cup B) = u(A) \cup u(B)$ for all $A, B \subset X$,

is called a *Čech closure operator* and the pair (X, u) is a *Čech closure space*. For short, (X, u) will be denoted by X as well, and called a *closure space* or a *space*.

A subset A is *closed* in (X, u) if $u(A) = A$ holds. It is *open* if its complement is closed.

The *interior operator* $\mathrm{int}_u : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by means of the closure operator in the usual way: $\mathrm{int}_u = \mathrm{c} \circ u \circ \mathrm{c}$, where $\mathrm{c} : \mathcal{P}(X) \to \mathcal{P}(X)$ is the complement operator. A subset U is a *neighbourhood* of a point x (respectively subset A) in X if $x \in \mathrm{int}_u U$ (respectively $A \subset \mathrm{int}_u U$) holds. The collection of all neighbourhoods of x will be denoted by $\mathcal{N}_x$ or $\mathcal{N}(x)$.

In (X, u), a point $x \in u(A)$ if and only if for each neighbourhood U of x, $U \cap A \neq \emptyset$ holds.

Separation axioms are defined in the usual way (see [3], Section 27). A space (X, u) is:

T_0 if for each two distinct points in X at least one has a neighbourhood which does not contain the other point.

T_1 if for each two distinct points in X the following holds: $(\{x\} \cap u(\{y\})) \cup (\{y\} \cap u(\{x\})) = \emptyset$ whenever $x \neq y$. It is equivalent to: every one-point subset of X is closed in (X, u).

T_2 (*Hausdorff*) if each two distinct points have disjoint neighbourhoods.

Regular if for each $x \notin u(A)$ there exist disjoint neighbourhoods U of x and V of A. It is equivalent to: for every point x and a neighbourhood U of x there is a neighbourhood V of x such that $x \in V \subset u(V) \subset U$ holds.

Normal if each two subsets A and B such that $u(A) \cap u(B) = \emptyset$, have disjoint neighbourhoods.

Since every completely regular (Tikhonov) Čech closure space is topological, we consider only spaces with lower separation axioms.

A function $f : (X, u) \to (Y, v)$ is *continuous* if for every $x \in X$ the inverse image of every neighbourhood of $f(x)$ is a neighbourhood of x; equivalently, $f(u(A)) \subset v(f(A))$ for every $A \subset X$. The set of all continuous functions from (X, u) to (Y, v) is denoted by $\mathcal{C}(X, Y)$.

A collection $\{G_\alpha\}$ is an *interior cover* of a set A in (X, u) if the collection $\{\mathrm{int}_u G_\alpha\}$ covers A. We suppose that the interior of every element of an interior cover is nonempty.

A subset A in a space (X, u) is *compact* if every interior cover of A has a finite subcover, not necessarily interior.

The *topological modification* $\hat{u}$ of the operator u is the finest Kuratowski closure operator coarser than u. The corresponding topology $\mathcal{T}(\hat{u})$ consists of all open sets in (X, u).

We also consider the topology $\mathcal{T}(\tilde{u})$ on (X, u) having for a basis the collection $\mathcal{J} = \{\mathrm{int}_v(B) \mid B \subset Y\}$. (See [1].) Its (Kuratowski) closure operator will be denoted by $\tilde{u}$.

On Y^X, the set of all functions from (X, u) to (Y, v), the product closure operator is defined by means of neighbourhoods: for every $f \in Y^X$ the family $\mathcal{N}(f) = \{\pi_x^{-1}(V) \mid x \in X,\ V \in \mathcal{N}_{f(x)} \text{ in } (Y, v)\}$, is a neighbourhood subbase at f. Canonical neighbourhoods are finite intersections of subbasis elements. Denote by Πv the product closure operator on Y^X. For subsets $B_x \subset Y$, we have the closure $(\Pi v)(\Pi_{x \in X} B_x) = \Pi_{x \in X} v(B_x)$ (the closure of the product of subsets is the product of the closures). Also, if G is a neighbourhood of $f \in Y^X$, then $\pi_x(G)$ is a neighbourhood of $f(x) \in Y$ and (Y, v) is homeomorphic to a subspace of the product.

As in the topological case, the product space $(Y^X, \Pi v)$ is T_0 (respectively T_1, T_2, regular) if and only if (Y, v) is such.

Also, the subspace of constant functions is homeomorphic to (Y, v).

All notions not explained here concerning Čech closure spaces can be found in [3] and [13]; while those concerning function spaces in [6] and [15].

3. Graph topologies for function spaces

Let (X, u) and (Y, v) be closure spaces, $(X \times Y, u \times v)$ their product closure space and $\mathcal{F} \subset Y^X$ a collection of functions from (X, u) to (Y, v). $G(f)$ is the usual notation for the graph of f.

Denote by $\mathcal{GO}$ the *graph-open* topology on $\mathcal{F}$ having for a basis sets of the form

$$[W] = \{f \mid G(f) \subset W\} \quad \text{and} \quad W \text{ is open in } (X \times Y, u \times v), \text{ that is } W \in \mathcal{T}(\widehat{u \times v}) = \mathcal{T}(\hat{u}) \times \mathcal{T}(\hat{v}); \tag{1}$$

and by $\mathcal{GI}$ the *graph-interior* topology on $\mathcal{F}$ having for a basis sets of the form

$$[\mathrm{int}\, W] = \{f \mid W \in \mathcal{N}(G(f))\} = \{f \mid G(f) \subset \mathrm{int}_{u \times v} W\}. \tag{2}$$

Clearly, $\mathcal{GO} \subset \mathcal{GI}$.

Theorem 3.1. *If (X, u) is a T_1-space and (Y, v) is a T_0-space, then $(\mathcal{F}, \mathcal{GI})$ is T_0.*

Proof. Let $f, g \in \mathcal{F}$ and $f \neq g$ hold. There is an $x \in X$ such that $f(x) \neq g(x)$. We can assume that there exists a $V \in \mathcal{N}(f(x))$ such that $g(x) \notin V$. Since (X, u) is T_1 the set $W = (X \backslash \{x\}) \times Y \cup (X \times V)$ is a neighbourhood of $G(f)$, while $G(g) \not\subset W$. Thus $f \in [\mathrm{int}\, W]$ and $g \notin [\mathrm{int}\, W]$.

Note that if (X, u) is T_1 and $(Y, \hat{v})$ is T_0, then $(\mathcal{F}, \mathcal{GO})$ is T_0. □

Conversely, we have

Theorem 3.2. *If $\mathcal{F}$ contains the constant functions and $(\mathcal{F}, \mathcal{GO})$ is T_0, then $(Y, \hat{v})$, and thus (Y, v), is T_0.*

Theorem 3.3. *For a set Y that contains at least two points the following holds*:

(i) *If (X, u) and (Y, v) are T_1-spaces then $(\mathcal{F}, \mathcal{GO})$ and $(\mathcal{F}, \mathcal{GI})$ are T_1;*
(ii) *If $\mathcal{F}$ contains the constant functions and $(\mathcal{F}, \mathcal{GO})$ or $(\mathcal{F}, \mathcal{GI})$ is T_1, then (X, u) and (Y, v) are T_1.*

Proof. (i) $\Rightarrow$ (ii) If (X, u) and (Y, v) are T_1, then for $f \neq g$ there is an $x \in X$ such that $f(x) \neq g(x)$. The set $W = (X\backslash\{x\}) \times Y \cup X \times (Y\backslash\{f(x)\})$ is an open neighbourhood of $G(f)$, while $G(g) \not\subset W$. It follows that $(\mathcal{F}, \mathcal{GO})$ is T_1 and so is $(\mathcal{F}, \mathcal{GI})$.

(ii) $\Rightarrow$ (i) Conversely, let $(\mathcal{F}, \mathcal{GI})$ be T_1. If (Y, v) is not T_1, equivalently $(Y, \hat{v})$ is not T_1, there are $y_1, y_2 \in Y$, $y_1 \neq y_2$, such that for every $V \in \mathcal{N}(y_1)$, $y_2 \in V$. We can take V to be open in (Y, v). Let f and g be constant functions, $f(X) = \{y_1\}$ and $g(X) = \{y_2\}$. Then for every $W \in \mathcal{N}(G(f))$, $G(g) \subset W$ holds since $(x, f(x)) = (x, y_1) \in U(x) \times V \subset W$ implies $(x, g(x)) = (x, y_2) \in U(x) \times V \subset W$, that is $f \in [\operatorname{int} W]$ implies $g \in [\operatorname{int} W]$ and $(\mathcal{F}, \mathcal{GI})$ is not T_1.

To prove that (X, u) is T_1, suppose it is not. There are $x_1, x_2 \in X$, $x_1 \neq x_2$, such that for every $U \in \mathcal{N}(x_1)$, $x_2 \in U$ holds. We can take U to be open in (X, u). Pick $y_1, y_2 \in Y$, $y_1 \neq y_2$, and set $g(x) = y_1$ for all x and $f(x) = y_1$ for $x \neq x_2$, and $f(x_2) = y_2$. For every $W \in \mathcal{N}(G(f))$, $W \in \mathcal{N}(G(g))$ holds, as for each $x \neq x_2$, $f(x) = g(x)$ and $(x_1, f(x_1)) = (x_1, y_1) \in U \times V(y_1) \subset W$ imply $(x_2, g(x_2)) = (x_2, y_1) \in U \times V(y_1) \subset W$. So $(\mathcal{F}, \mathcal{GI})$ is not T_1. $\square$

Theorem 3.4. *For a set Y that contains at least two points the following holds*:

(i) *If (X, u) is T_1 and (Y, v) is T_2, then $(\mathcal{F}, \mathcal{GI})$ is T_2;*
(ii) *If $\mathcal{F}$ contains the constant functions and $(\mathcal{F}, \mathcal{GO})$ is T_2, then (X, u) is T_1 and $(Y, \hat{v})$, and thus (Y, v), is T_2.*

Proof. If (X, u) is T_1 and (Y, v) is T_2, then for $f \neq g$ there is an $x \in X$ such that $f(x) \neq g(x)$. Since (Y, v) is T_2, there are $V_1 \in \mathcal{N}(f(x))$ and $V_2 \in \mathcal{N}(g(x))$ such that $V_1 \cap V_2 = \emptyset$. Since (X, u) is T_1, the set $X\backslash\{x\}$ is open in X, so $W_1 = (X\backslash\{x\}) \times Y \cup (X \times V_1)$ and $W_2 = (X\backslash\{x\}) \times Y \cup (X \times V_2)$ are disjoint neighbourhoods of $G(f)$, and $G(g)$ respectively. Hence $f \in [\operatorname{int} W_1]$, $g \in [\operatorname{int} W_2]$, and $[\operatorname{int} W_1] \cap [\operatorname{int} W_2] = \emptyset$. So $(\mathcal{F}, \mathcal{GI})$ is T_2.

Conversely, let $(\mathcal{F}, \mathcal{GO})$ be T_2. If $(Y, \hat{v})$ is not T_2, there are distinct y_1, y_2 such that for every open $V_1 \in \mathcal{N}(y_1)$ and $V_2 \in \mathcal{N}(y_2)$, $V_1 \cap V_2 \neq \emptyset$ holds. Let f and g be constant functions, $f(X) = \{y_1\}$ and $g(X) = \{y_2\}$. Then for every open $W_1 \in \mathcal{N}(G(f))$ and $W_2 \in \mathcal{N}(G(g))$, $W_1 \cap W_2 \neq \emptyset$ holds. Pick an $x \in X$. There are open sets U_1 and U_2 in (X, u) and V_1 and V_2 in (Y, v) such that $(x, f(x)) = (x, y_1) \in U_1 \times V_1 \subset W_1$ and $(x, g(x)) = (x, y_2) \in U_2 \times V_2 \subset W_2$. For any $y \in V_1 \cap V_2$ we have $(x, y) \in W_1 \cap W_2$.

That (X, u) is T_1, follows from the previous statement. $\square$

Theorem 3.5. *Let (X, u) be a regular space, $(Y, \hat{v})$ be T_0 and $\mathcal{F} \subset \mathcal{C}(X, Y)$. Then $(\mathcal{F}, \mathcal{GI})$ is T_0.*

Proof. Let $f \neq g$ in $\mathcal{F}$. There is an $x \in X$ such that $f(x) \neq g(x)$. We can assume that there exists an open set $V \subset Y$ such that $f(x) \in V$ and $g(x) \notin V$. Since f is continuous and X is regular, there is a $U \subset X$ such that $U \in \mathcal{N}(x)$ and $u(U) \subset f^{-1}(V)$. Then $G(f) \subset \operatorname{int}_{u \times v} W$ where $W = (f^{-1}(V) \times V) \cup (U^{c} \times Y)$, while $G(g) \not\subset W$. Hence $f \in [\operatorname{int} W]$ while $g \notin [\operatorname{int} W]$. $\square$

Lemma 1. *If $f : (X, u) \to (Y, v)$ is a continuous function and Y is a T_2-space, then the graph $G(f)$ is closed in $X \times Y$.*

Proof. If $(x, y) \notin G(f)$ then $y \neq f(x)$. Since Y is T_2, there are disjoint sets $V_1 \in \mathcal{N}(f(x))$, $V_2 \in \mathcal{N}(y)$. Since f is continuous, there is a $U \in \mathcal{N}(x)$ such that $f(U) \subset V_1$. It follows that $U \times V_2 \in \mathcal{N}(x, y)$ in $X \times Y$ and $U \times V_2 \subset G(f)^{c}$. Thus $G(f)^{c}$ is open being a neighbourhood of each of its points. $\square$

Lemma 2. *If $f : (X, u) \to (Y, v)$ is a continuous function and W an open subset in $X \times Y$, then the set $A = \{x \mid (x, f(x)) \in W\}$ is open in X.*

Proof. Analogous to that of Lemma 3.9 in [12]. $\square$

Theorem 3.6. *Let Y be a T_2-space and $\mathcal{F} \subset \mathcal{C}(X, Y)$. If $X \times Y$ is normal, then $(\mathcal{F}, \mathcal{GO})$ is regular.*

Proof. Let $[W]$ be a neighbourhood of f in $(\mathcal{F}, \mathcal{GO})$, that is W be an open set in $X \times Y$ containing $G(f)$. By Lemma 1, $G(f)$ is closed in $X \times Y$. Thus $G(f) \subset W$ implies that $G(f)$ and W^{c} have disjoint closures. By normality of $X \times Y$ there are disjoint $H_1, H_2 \subset X \times Y$, $H_1 \in \mathcal{N}(G(f))$, $H_2 \in \mathcal{N}(W^{c})$. To prove the statement, it is enough to show that $\mathcal{GO}\,\mathrm{cl}[\operatorname{int} H_1] \cap [W]^{c} = \emptyset$. Let $g \in [W]^{c}$. Then $G(g) \not\subset W$. The set $A = \{x \mid (x, g(x)) \in W\}$ is open in X by Lemma 2. Hence $G(g) \subset \operatorname{int}((A \times Y) \cup H_2)$.

Moreover, $[\operatorname{int} H_1] \cap [\operatorname{int}(A \times Y) \cup H_2] = \emptyset$ since $h \in [\operatorname{int} H_1] \cap [\operatorname{int}(A \times Y) \cup H_2]$ implies $G(h) \subset \operatorname{int}(H_1 \cap ((A \times Y) \cup H_2))$. Let $x \notin A$. Then $(x, h(x)) \in H_1 \cap H_2 = \emptyset$, which is a contradiction. □

4. Compact-open and compact-interior topology

Let (X, u) and be (Y, v) be closure spaces. For $A \subset X$ and $B \subset Y$ let

$$[A, B] = \{ f \in \mathcal{F} \mid f(A) \subset B \}. \tag{3}$$

The sets $[C, V]$ where C is a compact subset of X and V open in Y form a subbase for a topology $\mathcal{CO}$ on $\mathcal{F}$ which will be called the *compact-open topology*.

The sets $[C, \operatorname{int}_v V]$ where C is a compact subset of X and $V \subset Y$ form a subbase for a topology $\mathcal{CI}$ on $\mathcal{F}$ which will be called the *compact-interior topology*.

Note that in the case when (Y, v) is a topological space the compact-interior and the compact-open topologies coincide.

Theorem 4.1. *If (Y, v) is a T_2-space, then $(\mathcal{F}, \mathcal{CI})$ is T_2.*

Proof. Clear. □

Theorem 4.2. *If $(Y, \hat{v})$ is T_2 (respectively regular), then the space $(\mathcal{C}(X, Y), \mathcal{CO})$ is T_2 (respectively regular).*

Proof. We prove only regularity. Let $[C, V]$ be a subbase element and $f \in [C, V]$. By regularity of $(Y, \hat{v})$, for each $x \in C$ there is an open set W^x such that $f(x) \in W^x \subset \hat{v}(W^x) \subset V$. The collection $\{W^x \mid x \in C\}$ is an interior cover of the compact set $f(C)$, so there is a finite subcover $\{W^{x_i} \mid i = 1, \dots, m\}$. The set $W = \bigcup_{i=1}^m W^{x_i}$ is open and $f(C) \subset W \subset \hat{v}(W) \subset V$ holds.

Note that the set $\hat{v}(W)^c$ is open in (Y, v), hence for each $x \in C$ the set $[\{x\}, \hat{v}(W)^c]$ is open and, consequently, $[\{x\}, \hat{v}(W)] = [\{x\}, \hat{v}(W)^{cc}]$ is closed in $\mathcal{CO}$. It follows that $[C, \hat{v}(W)] = \bigcap_{x \in C} [\{x\}, \hat{v}(W)]$ is closed, and the open set $[C, W] \in \mathcal{CO}$ satisfies the inclusions $[C, W] \subset \mathcal{CO}\mathrm{cl}[C, W] \subset [C, \hat{v}(W)] \subset [C, V]$. □

Theorem 4.3. *If $(Y, \mathcal{V})$ is a locally compact T_2 topological space, then the composition map*

$$\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$$

is continuous, provided the compact-open topology is used throughout.

Proof. Let $(f, g) \in \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z)$ and $g \circ f \in [C, W]$, where C is compact in (X, u) and W open in (Z, w). Since $g(f(C)) \subset W$, for each $y \in f(C)$ there is an open set $G(y) \subset Y$ such that $g(G(y)) \subset W$. There is an open set $V(y)$ such that $y \in V(y) \subset \overline{V}(y) \subset G(y)$ and $\overline{V}(y)$ is compact. Consider a finite subcover $\{V(y_i) \mid i = 1, \dots, m\}$ of the compact set $f(C)$. Then $f(C) \subset V = \bigcup_{i=1}^m V(y_i) \subset \overline{V} = \bigcup_{i=1}^m \overline{V}(y_i) \subset \bigcup_{i=1}^m G(y_i)$. For each $f_1 \in [C, V]$ and $g_1 \in [\overline{V}, W]$, $(g_1 \circ f_1)(C) = g_1(f_1(C)) \subset g_1(V) \subset g_1(\overline{V}) \subset W$ holds. □

We recall the notions we use in the sequel. (See [2] or [13].)

Let X, Y and Z be three spaces. The *exponential function* is the mapping $E : \mathcal{C}(Z \times X, Y) \to \mathcal{C}(Z, \mathcal{C}(X, Y))$ defined as follows: for every (continuous) $g : Z \times X \to Y$ the function $E(g) = g^*$ from Z to $\mathcal{C}(X, Y)$ is defined by $g^*(z)(x) = g(z, x)$. Conversely, to each $g^* : Z \to \mathcal{C}(X, Y)$ the corresponding function $g : Z \times X \to Y$ defined by $g(z, x) = g^*(z)(x)$ is continuous in x for each fixed z.

The *evaluation mapping* $\varepsilon : \mathcal{C}(X, Y) \times X \to Y$ is defined by $\varepsilon(f, x) = f(x)$.

A topology $\mathcal{T}$ on $\mathcal{C}(X, Y)$ is called:

proper (*splitting*) if for any closure space (Z, w),

(1) $g : (Z, w) \times (X, u) \to (Y, v)$ is continuous $\Rightarrow g^* : (Z, w) \to (\mathcal{C}(X, Y), \mathcal{T})$ is continuous;

admissible (*jointly continuous*) if for every space (Z, w),

(2) $g^* : (Z, w) \to (\mathcal{C}(X, Y), \mathcal{T})$ is continuous $\Rightarrow g : (Z, w) \times (X, u) \to (Y, v)$ is continuous.

Theorem 4.4. *(Theorem 10 in [13]) The compact-open topology on $\mathcal{C}(X, Y)$ is proper.*

Theorem 4.5. *If the evaluation mapping $\varepsilon : (\mathcal{C}(X, Y), \mathcal{T}) \times (X, u) \to (Y, v)$ is continuous for some topology $\mathcal{T}$, then $\mathcal{T}$ is finer than the compact-open topology.*

Proof. By Theorem 1 in [13], the evaluation mapping $\varepsilon : (\mathcal{C}(X, Y), \mathcal{T}) \times (X, u) \to (Y, v)$ is continuous if and only if $\mathcal{T}$ is admissible. By Theorem 4.4, the compact-open topology $\mathcal{CO}$ is proper, and by Theorem 4 in [13], $\mathcal{T}$ is finer than $\mathcal{CO}$, i.e. $\mathcal{CO} \subset \mathcal{T}$ holds. □

We use the following

Definition. A closure space (X,u) is *locally compact* if compact neighbourhoods form a local base at each of its points.

Theorem 4.6. *If (X,u) is locally compact and $(Y,\mathcal{V})$ is a topological space, the space $\mathcal{C}(X,Y)$ with the compact-open topology is admissible.*

Proof. By Theorem 1 in [13] it is sufficient to prove that the evaluation map $\varepsilon : (\mathcal{C}(X,Y),\mathcal{T}) \times (X,u) \to (Y,v)$ is continuous. Let $(f,x) \in \mathcal{C}(X,Y) \times X$ and $V \in \mathcal{V}$ be an open neighbourhood of $\varepsilon(f,x) = f(x)$. By continuity of f there is an open neighbourhood U of x such that $f(U) \subset V$. Let C be a compact neighbourhood of x contained in U. Then $[C,V] \times C$ is a neighbourhood of (f,x) and for any $f_1 \in [C,V]$ and $x_1 \in C$, $\varepsilon(f_1,x_1) = f_1(x_1) \in f_1(C) \subset V$ holds. □

Theorems 4.4 and 4.6 generalize the well-known fact that for a regular locally compact topological space X and an arbitrary topological space Y, the compact-open topology on $\mathcal{C}(X,Y)$ is always proper and admissible (see Theorem 1 in [7] and Theorem 4.71 in [2]).

5. Relations between some topologies on Y^X and the product closure space

Denote by $\mathcal{TO}$ the topology on Y^X having for a subbase sets of the form

$$\big[\{x\},V\big] = \big\{f \bigm| f(x) \in V\big\} \quad \text{where } V \text{ is open in } (Y,v); \tag{4}$$

and by $\mathcal{TI}$ the topology on Y^X having for a subbase sets of the form

$$\big[\{x\},\operatorname{int}_v V\big] = \big\{f \bigm| V \text{ is a neighbourhood of } f(x) \text{ in } (Y,v)\big\} = \big\{f \bigm| V \in \mathcal{N}\big(f(x)\big)\big\}. \tag{5}$$

In other words, the collection $[\{x\},G]$ for $x \in X$ and $G \in \mathcal{J}$ is a subbase for the topology $\mathcal{TI}$.

The topology $\mathcal{TO}$ is the product of topological modifications, $\mathcal{TO} = \mathcal{T}(\Pi\hat{v}) = \mathcal{T}(\widehat{\Pi v})$ and $\mathcal{TI} = \mathcal{T}(\Pi\tilde{v}) = \mathcal{T}(\widetilde{\Pi v})$.

Hence on Y^X the following is true: the closure operator defined by $\mathcal{TO}$ is coarser than Πv, which is coarser than the closure operator defined by $\mathcal{TI}$.

By taking the singletons for the sets C in the definition of compact-open and compact-interior topology on Y^X, the topologies $\mathcal{TO}$ and $\mathcal{TI}$ are obtained. Hence $\mathcal{TO}$ is coarser than the compact-open topology $\mathcal{CO}$ and the topology $\mathcal{TI}$ is coarser than the compact-interior topology $\mathcal{CI}$.

Theorem 5.1.

(i) *The spaces $(Y^X,\mathcal{TI})$ and $(Y^X,\mathcal{CI})$ are T_0 (respectively T_1, T_2) if and only if (Y,v) is T_0 (respectively T_1, T_2).*
(ii) *The spaces $(Y^X,\mathcal{TO})$ and $(Y^X,\mathcal{CO})$ are T_0 (respectively T_1, T_2) if and only if $(Y,\hat{v})$ is T_0 (respectively T_1, T_2).*

Proof. (i) If (Y,v) is T_0 (respectively T_1, T_2), then the product space $(Y^X,\Pi v)$ is T_0 (respectively T_1, T_2), and so are $(Y^X,\mathcal{TI})$ and $(Y^X,\mathcal{CI})$.

To prove the converse, let $(Y^X,\mathcal{TI})$ be T_0, $y_1, y_2 \in Y$ be two distinct points and $c_{y_1}, c_{y_2} \in Y^X$ the corresponding constant functions. Let $\bigcap_{i=1}^k [\{x_i\},\operatorname{int}_v V_i]$ be a canonical neighbourhood of c_{y_1} in $(Y^X,\mathcal{TI})$ not containing c_{y_2}. Then $\bigcap_{i=1}^k V_i$ is a neighbourhood of y_1 in (Y,v) which does not contain y_2.

Now suppose that $(Y^X,\mathcal{CI})$ is T_2 and $c_{y_1}, c_{y_2} \in Y^X$ are two distinct elements with disjoint canonical neighbourhoods $\bigcap_{i=1}^k [C_i',\operatorname{int}_v V_i']$ and $\bigcap_{j=1}^m [C_j'',\operatorname{int}_v V_j'']$ respectively. The sets $V' = \bigcap_{i=1}^k V_i'$ and $V'' = \bigcap_{j=1}^m V_j''$ are disjoint neighbourhoods in (Y,v) of y_1 and y_2 respectively.

The other properties may be proved in a similar way. □

Theorem 5.2. *Denote by $K = \{c_y \mid y \in Y\}$ the set of all constant functions from (X,u) into (Y,v). Then:*

(i) *In $(Y^X,\mathcal{TO})$ and $(Y^X,\mathcal{CO})$ the subspace K is homeomorphic to $(Y,\hat{v})$;*
(ii) *In $(Y^X,\mathcal{TI})$ and $(Y^X,\mathcal{CI})$ the subspace K is homeomorphic to $(Y,\tilde{v})$.*

6. Relations between graph topology and other function space topologies

Theorem 6.1. *If (X,u) is a T_1-space, then the closure operator defined by $\mathcal{TO}$ is coarser than Πv, which is coarser than the closure operator defined by $\mathcal{TI}$ and $\mathcal{TI} \subset \mathcal{GI}$ holds.*

Proof. For the last inclusion, a subbasis element in $\mathcal{TI}$ is of the form $[\{x\}, \mathrm{int}_v V] = \{f \mid V \in \mathcal{N}(f(x))\}$. Since X is T_1, the set $X\backslash\{x\}$ is open in X. For the set $W = (X\backslash\{x\}) \times Y \cup (X \times V)$ we have $[\mathrm{int}\, W] = \{f \mid f(x) \in \mathrm{int}_v V\} = [\{x\}, \mathrm{int}_v V]$.

By a similar argument it may be proved that $\mathcal{TO} \subset \mathcal{GO}$ holds. □

Theorem 6.2.

(i) *If (X, u) is a T_2-space, then $\mathcal{CI} \subset \mathcal{GI}$ and $\mathcal{CO} \subset \mathcal{GO}$.*
(ii) *If $(X, \hat{u})$ is compact and T_2, then the graph-open topology coincides with the compact-open topology.*

Proof. (i) A subbasis element in $\mathcal{CI}$ is of the form $[C, \mathrm{int}_v V] = \{f \mid V \in \mathcal{N}(f(x))\}$ for every $x \in C$. Since X is T_2, the set $X\backslash C$ is open in X. For the set $W = ((X\backslash C) \times Y) \cup (X \times V)$ we have $[\mathrm{int}\, W] = \{f \mid f(C) \subset \mathrm{int}_v V\} = [C, \mathrm{int}_v V]$.

In the case when the topology $\mathcal{CO}$ is considered, the sets V and W are open.

(ii) Follows easily from the definitions. □

Theorem 6.3. *Let (X, u) or $(X, \hat{u})$ be a regular space, $(Y, \mathcal{V})$ be a topological space and $\mathcal{F} \subset \mathcal{C}(X, Y)$. Then the compact-open topology on $\mathcal{F}$ is contained in the graph-open topology.*

Proof. Suppose (X, u) is regular. A subbasis element in $\mathcal{CO}$ is of the form $[C, V]$ where $V \in \mathcal{V}$. Since X is regular, for each $x \in C$ there is a neighbourhood $U(x)$ such that $u(U(x)) \subset f^{-1}(V)$. The collection $\{U(x) \mid x \in C\}$ is an interior cover of the compact set C, so there is a finite subcover $\{U(x_i) \mid i = 1, \ldots, m\}$. Set $U = \bigcup_{i=1}^{m} U(x_i)$. The set $W = (f^{-1}(V) \times V) \cup (U^{c} \times Y)$ is a neighbourhood of $G(f)$ since for $x \in u(U)$, $f^{-1}(V) \times V$ is an open neighbourhood of $(x, f(x))$, while $U^{c} \times Y \in \mathcal{N}(x, f(x))$ for $x \in u(U)^{c}$.

That $[C, V] = [W]$ follows from the fact that for each $g \in [W]$ and $x \in C$, $(x, g(x)) \in G(g) \subset W$ implies $(x, g(x)) \in f^{-1}(V) \times V$, so $g(C) \subset V$, i.e. $g \in [C, V]$ holds.

(ii) By a similar argument, if $(X, \hat{u})$ is regular, for a compact set C and open V, C is compact in $(X, \hat{u})$, so by regularity there is an open set U in X such that $C \subset U \subset \mathrm{cl}_{\hat{u}}(U) \subset f^{-1}(V)$ holds. The set $W = (f^{-1}(V) \times V) \cup (\hat{u}(U)^{c} \times Y)$ is an open neighbourhood of $G(f)$ and $g \in [C, V]$ holds. □

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